NOTES ON MECHANICS

PREPARED FOR

THE USE OF THE STUDENTS

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THE PENNSYLVANIA STATE COLLEGE

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Parallel Forces in Space.—Refer the forces to a set of rectangular co-ordinate axes such that the Z axis is parallel to the forces.

Denote the forces by P_1, P_2, P_3, etc., and their co-ordinates by x_1y_1, x_2y_2, etc., and the resultant by R and its co-ordinates by XY.

Then, the magnitude of the resultant is equal to the algebraic sum of the forces, or $R = \Sigma P$.

The moment of the resultant about OY is equal to the sum of the moments of the forces about OY, or $X\Sigma P = \Sigma Px$.

For a similar reason, $Y\Sigma P = \Sigma Py$.

Hence, $X = \frac{\Sigma Px}{\Sigma P}$ and $Y = \frac{\Sigma Py}{\Sigma P}$.

It may also be proven, that in the case where the forces are not parallel to either of the rectangular axes,

$$R = \Sigma P, \quad X = \frac{\Sigma Px}{\Sigma P}, \quad Y = \frac{\Sigma Py}{\Sigma P}, \quad Z = \frac{\Sigma Pz}{\Sigma P},$$

X, Y, and Z being the co-ordinates of a point in the line of action of R.

Note.—In this case the products Px, Py, Pz, etc., are not moments.

Centre of Gravity.—One particular case of parallel forces is found in the attraction exerted by the force of gravity upon the different particles of any body. For convenience, we speak of the

center of gravity of surfaces and of lines, although such surfaces and lines have no mass, meaning by that term the point of application of the resultant force which would be in equilibrium with a system of parallel forces uniformly distributed over such surface or along such line. This leads us to the following definition. The center of gravity of a body, surface, or line is the point of application of a resultant force which will hold in equilibrium a system of parallel and uniformly distributed forces acting upon the body, surface, or line.

If X, Y, Z denote the co-ordinates of the center of gravity of any body referred to three co-ordinate axes and if we denote the volume of an element by dV and its heaviness or weight per unit volume by w, and the co-ordinates of the element by x, y, z; then using the integral sign to indicate the summation of like terms for all particles of the body, we have, for heterogeneous bodies,

$$X=\frac{\int wxdV}{\int wdV}, \quad Y=\frac{\int wydV}{\int wdV}, \quad Z=\frac{\int wzdV}{\int wdV}.$$

If the body is homogeneous, w is constant and cancels out leaving

$$X=\frac{\int xdV}{V}, \quad Y=\frac{\int ydV}{V}, \quad Z=\frac{\int zdV}{V}.$$

We may consider a surface to be an infinitely thin homogeneous shell of uniform thickness and if dA denote an element and A the whole area of the surface, the above equations become

$$X=\frac{\int xdA}{A}, \quad Y=\frac{\int ydA}{A}, \quad Z=\frac{\int zdA}{A}.$$

Similarly for a homogeneous wire of constant, small cross-sec-

tion, and for its limit, a geometric line, its length being s and an element of length, ds, we obtain

$$X=\frac{\int xds}{s},\quad Y=\frac{\int yds}{s},\quad Z=\frac{\int zds}{s}.$$

PROBLEM 1.—Required the position of the center of gravity of a circular arc AB. Fig. 1. Take the origin O at the center of the circle, and the X axis bisecting the arc. Let the length of the arc be s and let ds denote an element of the arc. We need determine only X, since Y=0 from the conditions of symmetry.

$$X=\frac{\int xds}{s}.$$

From similar triangles we have

$$ds : dy :: r : x,$$

therefore $ds=\frac{rdy}{x}$, and $X=\frac{r}{s}\int_{-a}^{+a} dy=\frac{2ra}{s}$, i. e., equals the chord $\times$ the radius $\div$ the length of the arc. For a semicircular arc this reduces to $X=2r\div\pi$.

PROBLEM 2.—Center of gravity of trapezoids and triangles. Fig. 2. Prolong the non-parallel sides of the trapezoid to intersect at O, which take as the origin, making the X axis perpendicular to the bases b and b_1. Taking a vertical strip as our element of area, we have for its height, $\frac{b}{h}x$, and for its area, $dA=\frac{b}{h}xdx$.

Now $A=\frac{1}{2}(bh-b_1h_1)=\frac{1}{2}\frac{b}{h}(h^2-h_1^2)$ and $X=\frac{\int xdA}{A}$ becomes

$$X=\frac{\frac{b}{h}\int_{h_1}^{h} x^2dx}{\frac{1}{2}\frac{b}{h}(h^2-h_1^2)}=\frac{2}{3}\,\frac{h^3-h_1^3}{h^2-h_1^2}$$

for the trapezoid.

For a triangle $h_1=0$, and we have $X=\frac{2}{3}h$; that is, the center of gravity of a triangle is one third the altitude from the base. The center of gravity is finally determined by knowing that a line joining the middles of the parallel sides of the trapezoid, or the vertex to the center of the base of a triangle, passes through the center of gravity.

PROBLEM 3.—Sector of a circle. Fig. 3. Let the angle of the sector $=2\alpha$. Using polar co-ordinates, the element of area $dA=\rho d\rho . d\phi$, and its $x=\rho\cos\phi$; hence the total area

$$A=\int_{-\alpha}^{+\alpha}\left\{\int_0^r \rho\, d\rho\right\} d\phi=\int_{-\alpha}^{+\alpha}\frac{1}{2}r^2\, d\phi=r^2\alpha.$$

$$X=\frac{1}{A}\int x\,dA$$

$$=\frac{1}{A}\int_{-\alpha}^{+\alpha}\int_0^r \cos\phi\,\rho^2\, d\rho\, d\phi$$

$$=\frac{1}{A}\frac{r^3}{3}\int_{-\alpha}^{+\alpha}\cos\phi\, d\phi$$

$$=\frac{2}{3}\frac{r\sin\alpha}{\alpha}$$

or, putting $\beta=2\alpha$, $X=\frac{4}{3}\frac{r\sin\frac{1}{2}\beta}{\beta}$.

For a semicircle this reduces to $X=\frac{4r}{3\pi}$.

PROBLEM 4.—Sector of a flat ring, or annulus. The treatment is similar to that of the last case, the difference being that the limits of integration are $\Big|_{r_2}^{r_1}$ instead of $\Big|_{0}^{r}$, giving as the result

$$X=\frac{4}{3}\frac{r_1^3-r_2^3}{r_1^2-r_2^2}\frac{\sin\frac{1}{2}\beta}{\beta}.$$

PROBLEM 5.—Segment of a circle. Fig. 4. Since each rectangular element of any vertical strip has the same x, we may

take these vertical strips as the elementary areas. Then $dA=2ydx$. From similar triangles $x : y :: dy : dx$, i. e., $xdx=ydy$.

$$\text{Hence, } X=\frac{\int xdA}{A}=\frac{\int x2ydx}{A}=\frac{2\int_0^a y^2dy}{A}=\frac{2}{3}\,\frac{a^3}{A}$$

but $a =$ the half chord, hence, $X=\frac{(\text{chord})^3}{12A}$.

PROBLEM 6.—Homogeneous cone or pyramid. Divide the body into laminae by means of planes parallel to the base and apply the principles of the preceding cases.

Centrobaric Method of Determining Areas and Volumes.—If an elementary area, dA, be revolved about an axis in its plane, but not included in its area, through the angle, α, not greater than 2π, the distance of the elementary area from the axis of revolution being x, the volume generated is $dV=\alpha xdA$, and the total volume generated by all the elementary areas of a finite plane figure which lies entirely on one side of the axis of revolution and whose plane contains that axis is

$$V=\int dV=\alpha\int xdA;$$

$$\text{but } \alpha\int xdA=\alpha AX, \quad \text{and hence}$$

αX being the length of the path described by the center of gravity of the plane figure, we may write: The volume of a solid of revolution generated by a plane figure, lying on one side of the axis, equals the area of the figure multiplied by the length of the curve described by the center of gravity of the plane figure.

A similar proposition may be deduced for the surface generated by the revolution of a line.

The angle, α, must be expressed in radians for numerical work.

Non-concurrent Forces in Space.—Fig. 5. Let P_1, P_2, etc., be the given forces, and $x_1\, y_1\, z_1$, etc., the co-ordinates of their points of application referred to an arbitrary origin and axes;

α_1 β_1 γ_1, etc., the angles made by their lines of action with X, Y, and Z.

Considering the first force, P_1, replace it by its three components parallel to the three axes. $X_1 = P_1 \cos\alpha_1$, $Y_1 = P_1 \cos\beta_1$, $Z_1 = P_1 \cos\gamma_1$. P_1 itself is not shown in the figure. At O and also at A put a pair of equal and opposite forces, each equal and parallel to Z_1. Z_1 at B is now replaced by the single force Z_1 acting upward at the origin, and two couples, one in a plane parallel to YZ and having a moment equal to $-Z_1y_1$, and the other in a plane parallel to XZ and having a moment equal to $+Z_1x_1$. Similarly at O and C put in pairs of forces equal and parallel to X_1 and we have X_1 at B replaced by the force X_1 at the origin and two couples, one in a plane parallel to XY and having a moment $+X_1y_1$, the other in a plane parallel to XZ and having a moment equal to $-X_1z_1$; and finally by a similar device, Y_1 at B is replaced by the force Y_1 at the origin and two couples parallel to XY and YZ and having moments, $-Y_1x_1$ and $+Y_1z_1$, respectively.

We have therefore replaced the force P_1 by three forces X_1, Y_1, and Z_1 at O and six couples. Combining each pair of couples whose axes are parallel, they can be reduced to three, viz.;

One with X axis and a moment equal to $Y_1z_1 - Z_1y_1$;

One with Y axis and a moment equal to $Z_1x_1 - X_1z_1$;

One with Z axis and a moment equal to $X_1y_1 - Y_1x_1$.

Dealing with each of the other forces, P_2, P_3, etc., in the same manner, the whole system may be finally replaced by three forces, ΣX, ΣY, ΣZ, at the origin and three couples whose moments are respectively,

$L = \Sigma(Yz - Zy)$ with its axis parallel to X;

$M = \Sigma(Zx - Xz)$ with its axis parallel to Y;

$N = \Sigma(Xy - Yx)$ with its axis parallel to Z.

The axes of these couples being parallel to the respective coordinate axes, X, Y, and Z, and proportional to the moments, L, M, and N respectively, the axis of their resultant whose moment is

G, must be the diagonal of a parallelopiped constructed on the three component axes proportional to L, M, and N. Therefore $G = \sqrt{L^2+M^2+N^2}$ and the resultant of ΣX, ΣY, and ΣZ is $R = \sqrt{(\Sigma X)^2+(\Sigma Y)^2+(\Sigma Z)^2}$ acting through the origin.

If α, β, γ are the direction angles of R, we have

$$\cos\alpha = \frac{\Sigma X}{R}, \quad \cos\beta = \frac{\Sigma Y}{R}, \quad \cos\gamma = \frac{\Sigma Z}{R};$$

and if λ, μ, ν, are the direction angles of the resultant couple G, we have

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}.$$

For equilibrium, $G=0$ and $R=0$, or stated separately, $\Sigma X=0$, $\Sigma Y=0$, $\Sigma Z=0$, $L=0$, $M=0$, and $N=0$. Hence, if a system of non-concurrent forces in space is in equilibrium, the plane systems formed by projecting the given system upon each of three arbitrary co-ordinate planes will each be in equilibrium.

Statics of Flexible Cords.—In what follows, all cords are supposed to be perfectly flexible and inextensible and all forces are supposed to be in the same plane. All of the deductions are based upon the following axioms:

First, the strain on any cord at any point can act only along the cord, or along the tangent if the cord be curved.

Second, we may apply to flexible cords in equilibrium all the conditions for equilibrium of rigid bodies.

Third, conditions of equilibrium can not be applied unless the system can be considered a free body. This is allowable only when we conceive the supports to be removed and the reactions exerted by the supports or fastenings put in.

These reactions having been put in, let us consider the case shown in Fig. 6. If we take any point, A, on the cord as a center of moments, knowing that the resultant, R, of the forces, P_1, P_2, P_3, situated on one side of A must act along the cord through A, we have

$P_1a_1 - P_2a_2 - P_3a_3 = R \times 0 = 0$. That is, in a system of cords in equilibrium in a plane, if a center of moments be taken on the cord, the algebraic sum of the moments of those forces situated on either side of this point will equal zero.

Pulley.—A cord in equilibrium over a pulley whose axle is smooth, has the same tension on both sides. Fig. 7. For, considering the pulley and its portion of cord free, $\Sigma Pa = 0$ about the center gives $P'r = Pr$, i. e., $P' = P$ = tension in the cord. Hence the pressure, R, at the axle bisects the angle, α, and therefore, if a weighted pulley rides upon a cord, ABC, Fig. 8, its position of equilibrium, B, may be found by cutting the vertical through A by an arc of radius, CD, equal to the length of the cord and center at C and then drawing a horizontal through the middle of AD cutting CD in B.

Weights Suspended by Fixed Knots.—Given all the geometric elements of Fig. 9 and one weight, G_1, required the remaining weights and the forces, H_o, V_o, H_n, V_n, at the points of support that equilibrium may exist. H_o and V_o are the horizontal and vertical components of the tension in the cord at O, and similarly H_n and V_n, those at n. There are n+2 unknowns. From the second axiom we have

$$\Sigma X = 0, \quad \Sigma Y = 0, \text{ that is, } H_o - H_n = 0 \text{ and}$$
$$(G_1 + G_2 + \cdots) - (V_o + V_n) = 0.$$

While from the third axiom, taking the successive knots, 1, 2, etc., as centers of moments, we have

$$-V_o x_1 + H_o y_1 = 0,$$
$$-V_o x_2 + H_o y_2 + G_1(x_2 - x_1) = 0,$$
$$-V_o x_3 + H_o y_3 + G_1(x_3 - x_1) + G_2(x_3 - x_2) = 0,$$

etc., for the n knots. Thus we have n+2 independent equations and can solve for the unknown quantities.

Loaded Cord as a Parabola.—If the weights are equal and infinitely small and are uniformly spaced along the horizontal, when

equilibrium exists, the cord, having no weight, will form a parabola. In Fig. 10, let w equal the weight of the loads per linear unit; let O be the vertex of the curve and M any point on the curve. We may consider the portion OM as a free body if the reactions of the contiguous portions of the cord are put in, H_0 and T, and these according to the first axiom must act along the tangent to the curve at O and M respectively. That is, H_0 is horizontal and T makes some angle, ϕ, whose tangent equals $\frac{dy}{dx}$, with the axis X. Applying the second axiom, $\Sigma X=0$ gives

$$T\cos\phi - H_0 = 0, \text{ i. e., } T\frac{dx}{ds} = H_0. \qquad (1)$$

$$\Sigma Y=0 \text{ gives } T\sin\phi - wx = 0, \text{ i. e., } T\frac{dy}{ds} = wx. \qquad (2)$$

Dividing (2) by (1), member by member, we have $\frac{dy}{dx} = \frac{wx}{H_0}$.

Therefore $dy = wx\frac{dx}{H_0}$ is the differential equation of the curve.

$$y = \frac{w}{H_0}\int_0^x xdx = \frac{wx^2}{2H_0}$$

$$\text{or,} \quad x^2 = 2H_0\frac{y}{w}$$

which is the equation of a parabola whose vertex is at O and whose axis is vertical.

Note.—The equation, $\frac{dy}{dx} = \frac{wx}{H_0}$, may also be obtained by considering that we have a free body, Fig. 11, acted on by three forces, T, H_0, and $R = wx$ acting vertically through the middle of the abscissa, x. The resultant of H_0 and R must be equal and opposite to T. Therefore $\tan\phi = \frac{R}{H_0}$ or $\frac{dy}{dx} = \frac{wx}{H_0}$. Evidently also, the tangent line bisects the abscissa, x, which is a property of the parabola.

The Catenary.—A flexible, inextensible cord or chain of uniform weight per unit of length hung at two points and supporting its weight alone forms a curve called a catenary. Let the tension, H_0, at the lowest point or vertex be represented, for convenience, by the weight of an imaginary length, c, of similar cord weighing w pounds per unit of length, i. e., $H_0=wc$. A portion of cord of length, s, weighs ws pounds. Fig. 12 shows as free and in equilibrium a portion of the cord of any length, s, measured from the vertex. The load is placed uniformly along the curve and not as in the last section.

$\Sigma Y=0$ gives $T\frac{dy}{ds}=ws$. $\Sigma X=0$ gives $T\frac{dx}{ds}=wc$. Hence, by division, $cdy=sdx$ and squaring, $c^2dy^2=s^2dx^2$. (1)

Put $dy^2=ds^2-dx^2$ and we have, after solving for dx,

$$dx=\frac{cds}{\sqrt{s^2+c^2}}; \quad \text{therefore } x=c\int_0^s\frac{ds}{\sqrt{s^2+c^2}}$$

and $$x=c\log\frac{s+\sqrt{s^2+c^2}}{c} \qquad (2)$$

which shows the relation between the horizontal projection and the length of the curve.

Again in (1) put $dx^2=ds^2-dy^2$ and solve for dy.

This gives $dy=\frac{sds}{\sqrt{c^2+s^2}}=\frac{1}{2}\frac{d(c^2+s^2)}{(c^2+s^2)^{1/2}}$.

Therefore $y=\frac{1}{2}\int_0^s(c^2+s^2)^{-1/2}d(c^2+s^2)$

and finally $y=\sqrt{s^2+c^2}-c$. (3)

Solving for c, we have $c=(s^2-y^2)\div 2y$. (4)

Direct Central Impact.—Suppose two masses, m_1 and m_2, to be moving in the same straight line so that the distance between them is constantly diminishing, and that when collision or impact occurs, the line of pressure between the two bodies coincides with the line connecting their centers of mass. Such a

collision is called direct central impact and the motion of each mass during contact is variably accelerated and rectilinear, the only force acting upon it being the pressure of the other body. While the bodies are in contact, the pressure between them gradually increases and the bodies are compressed, their centers of mass approaching each other until the pressure and consequent compression attain a maximum, at which instant the two bodies move with a common velocity. After this, if the bodies possess any elasticity, the pressure continues, but gradually reduces to zero when the contact ceases and the bodies separate with different velocities.

Reckoning time from the first instant of contact, let t' equal the duration of the first period of contact, and t'' that of the first plus the second. Let m_1 and m_2, Fig. 13, be the masses and v_1 and v_2, simultaneous velocities of the two centers of mass at any instant during contact. Let P be the variable pressure between the two bodies. At any instant the acceleration of m_1 is $f_1 = -P \div m_1$, and that of m_2 is $f_2 = +P \div m_2$, m_1 being retarded and m_2 accelerated in velocity. In general, $f = \frac{dv}{dt}$.

Therefore, $m_1 dv_1 = -Pdt$ and $m_2 dv_2 = +Pdt$. (1)

Summing all similar terms for the first part of the impact, letting the velocities before impact be u_1 and u_2 and the common velocity at the instant of maximum pressure, w, we have

$$m_1 \int_{u_1}^{w} dv_1 = -\int_{0}^{t'} Pdt, \text{ or } m_1(w-u_1) = -\int_{0}^{t'} Pdt. \qquad (2)$$

$$m_2 \int_{u_2}^{w} dv_2 = +\int_{0}^{t'} Pdt, \text{ or } m_2(w-u_2) = +\int_{0}^{t'} Pdt. \qquad (3)$$

Eliminating the second members of (2) and (3) and solving for w, we obtain

$$w = \frac{m_1 u_1 + m_2 u_2}{m_1 + m_2} \qquad (4)$$

which is the common velocity at the instant of maximum pressure.

If the impact is inelastic, the bodies do not separate but continue to move with a common velocity.

Suppose that the impact is partially elastic; that the bodies are of the same material; and that the summation $\int_{t'}^{t''} Pdt$ for the second period of impact bears a ratio, e, to that of $\int_{0}^{t'} Pdt$ already used. If the impact is not too severe, we have, summing equation (1) for the second period and letting v_1' and v_2' equal the velocities after impact,

$$m_1 \int_{w}^{v_1'} dv_1 = -\int_{t'}^{t''} Pdt, \text{ i. e., } m_1(v_1' - w) = -e \int_{0}^{t'} Pdt; \quad (5)$$

$$\text{and } m_2 \int_{w}^{v_2'} dv_2 = +\int_{t'}^{t''} Pdt, \text{ i. e., } m_2(v_2' - w) = +e \int_{0}^{t'} Pdt. \quad (6)$$

e is called the coefficient of restitution.

Having determined the value of $\int_{0}^{t'} Pdt$ from (2) and (3) in terms of the mass and initial velocities, substitute it and that of w from (4) in (5) and (6) and we have for the final velocities,

$$v_1' = \frac{m_1u_1 + m_2u_2 - em_2(u_1 - u_2)}{m_1 + m_2} \quad (7)$$

$$\text{and } v_2' = \frac{m_1u_1 + m_2u_2 + em_1(u_1 - u_2)}{m_1 + m_2}. \quad (8)$$

If e=0, i. e., for inelastic impact, $v_1' = v_2' = w$ in equation (4). If e=1, i. e., for elastic impact, (7) and (8) become somewhat simplified.

To determine e experimentally, let a ball of the substance fall upon a very large slab of the same substance, noting both the height of the fall, h, and the height of the rebound, H; then re-

garding the mass of the slab as infinite compared with the mass of the ball, we obtain $e=\sqrt{\frac{H}{h}}$.

Virtual Velocities.—If a material point is moving in any direction not coincident with that of the resultant force acting, and any element of its path, ds, be projected upon the line of action of the force, the length of this projection, du, is called the virtual velocity of the force. The product of a force by its virtual velocity is called its virtual moment or virtual work and is reckoned positive or negative according as the direction of the virtual velocity is the same as that of the force or not.

Proposition 1.—The virtual work of a force equals the algebraic sum of the virtual works of its components. Fig. 14. Take the direction of ds as the axis X; let P_1 and P_2 be the components of P; α_1, α_2, α, their angles with X. Then $P\cos\alpha = P_1 \cos\alpha_1 + P_2 \cos\alpha_2$.

Hence, $P\,ds\cos\alpha = P_1\,sd\cos\alpha_1 + P_2\,ds\cos\alpha_2$; but $ds\cos\alpha$ equals the projection of ds upon P, i. e., equals its virtual velocity, du. Therefore, $P\,du = P_1\,du_1 + P_2\,du_2$. If in Fig. 14, α_1 were greater than 90°, evidently we would have $P\,du = -P_1\,du_1 + P_2\,du_2$, i. e., $P_1\,du_1$ would then be negative and OD_1 would fall behind O. Hence the definition of positive and negative as above given.

This proof is equally applicable whether all the forces are in the same plane or not.

Proposition 2.—The sum of the virtual works equals zero for concurrent forces in equilibrium. The resultant force is zero, hence from the preceding proposition, $\Sigma(Pdu)=0$.

Proposition 3.—The sum of the virtual works equals zero for any small displacement or motion of a rigid body in equilibrium under non-concurrent forces in a plane, all points of the body moving parallel to this plane.

1st. Let the motion be a translation, all points of the body describing equal and parallel lengths equal to ds. Fig. 15. Take

X parallel to ds; let α_1, etc., be the angles of the forces with X. Then $\Sigma(P \cos \alpha) = 0$. Therefore ds $\Sigma(P \cos \alpha) = 0$, but ds cos $\alpha_1 = du_1$; ds cos $\alpha_2 = du_2$; etc. Therefore $\Sigma(Pdu) = 0$.

2nd. Let the motion be a rotation through a small angle, $d\theta$, in the plane of the forces about any point, O, in that plane. Fig. 16. With O as a pole let ρ_1 be the radius vector of the point of application of P_1 and a_1 its lever arm from O. Similarly for the other forces. In the rotation, each point of application describes a small arc, ds_1, ds_2 etc., proportional to ρ_1, ρ_2, etc., since $ds_1 = \rho_1 d\theta$, $ds_2 = \rho_2 d\theta$, etc.

$$P_1 a_1 + P_2 a_2 + \quad \ldots . = 0,$$

but from similar triangles, $ds_1 : du_1 :: \rho_1 : a_1$,

$$\text{therefore } a_1 = \frac{\rho_1 du_1}{ds_1} = \frac{du_1}{d\theta}.$$

Similarly $a_2 = \dfrac{du_2}{d\theta}$, etc.

Hence, we must have $\dfrac{P_1 du_1 + P_2 du_2 + \ldots}{d\theta} = 0$, i. e., $\Sigma(Pdu) = 0$.

Now since any small displacement or motion of a body may be conceived to be accomplished by a small translation followed by a rotation through a small angle, and since the foregoing deals only with the projection of paths, the proposition is established and is called the principle of virtual velocities.

Generality of the Principle of Virtual Velocities.— If any mechanism of flexible, inextensible cords, or of rigid bodies joined together, or of both, at rest, or in motion with very small accelerations, be considered collectively, or any portion of it, and all external forces be put in, then, disregarding friction, for a small portion of its prescribed motion $\Sigma(Pdu)$ must equal zero.

When the acceleration of the parts of the mechanism is not practically zero, $\Sigma(Pdu)$ will not equal zero, but some function of the mass and velocities.

MOMENT OF INERTIA.

Rigid Bodies.—The moment of inertia of a rigid body about any axis is the limit of the sum of the products of the masses of the elementary particles of which the body is composed by the squares of their distances from the axis.

If we let dM represent any element of mass, and ρ its distance from the axis, then the moment of inertia is

$$I = \int \rho^2 dM.$$

If we conceive the total mass of the body to be concentrated at a single point at a distance, k, from the axis, the moment of inertia becomes Mk^2, M being the total mass.

$$\text{Hence, } k^2 = \frac{I}{M}.$$

k is called the radius of gyration.

Plane Figures.—The moment of inertia of a plane figure is the limit of the sum of the products of the elements of area by the squares of their distances from the axis.

That is, if dA is any element of area, and x is its distance from the axis, then the moment of inertia

$$I = \int x^2 dA = Ak^2.$$

$$k^2 = \frac{I}{A}.$$

Two Parallel Axes.—Fig. 17. Let Z and Z′ be two parallel axes. Then $I_z = \int \rho^2 dM$ and $I_z' = \int \rho'^2 dM$. But d being the distance between the axes, and a and b, the co-ordinates of Z′ referred to O, $d^2 = a^2 + b^2$, and

$$\rho^2 = (x-a)^2 + (y-b)^2 = x^2 + y^2 + d^2 - 2ax - 2by.$$

Therefore $I_z' = \int \rho^2 dM + d^2 \int dM - 2a \int x dM - 2b \int y dM.$ (1)

But $\int \rho^2 dM = I_z$, $\int dM = M$, and from the theory of the center

of gravity, we have $\int xdM = MX$ and $\int ydM = MY$.

Therefore, $I_Z' = I_Z + M(d^2 - 2aX - 2bY)$ (2)

in which X and Y are the co-ordinates of the center of gravity of the body.

If Z is a gravity axis, both X and Y become equal to zero, and (2) becomes $I_Z' = I_g + Md^2$ or $k^2{}_Z' = k^2{}_g + d^2$. (3)

It is therefore evident that the moment of inertia about a gravity axis is less than that about any parallel axis.

If instead of the element of mass, we use the element of area, we obtain for plane figures, $I_Z' = I_g + Ad^2$. (4)

Two Sets of Rectangular Axes with the Same Origin.—For two sets of rectangular axes having the same origin e may deduce the following relations.

Fig. 18. Since $I_X = \int y^2dA$, and $I_Y = \int x^2dA$, we have

$$I_X + I_Y = \int (x^2 + y^2)dA.$$

Similarly, $I_U + I_V = \int (v^2 + u^2)dA$.

But since the x and y of any dA have the same hypothenuse as the u and v, we have $v^2 + u^2 = x^2 + y^2$. Therefore, $I_X + I_Y = I_U + I_V$.

Fig. 19. Let X be an axis of symmetry; then, given I_X and I_Y, O being anywhere on X, required I_U, U being an axis through O and making any angle, α, with X.

$$I_U = \int v^2dA = \int (y \cos \alpha - x \sin \alpha)^2 dA$$

$$\therefore I_U = \cos^2\alpha \int y^2dA - 2\sin\alpha \cos\alpha \int xydA + \sin^2\alpha \int x^2dA.$$

But since the area is symmetrical about X, in summing up the products xydA, for every term $x(+y)dA$, there is also a term $x(-y)dA$ to cancel it; therefore $\int xydA = 0$.

$$\text{Hence } I_U = \cos^2\alpha\, I_X + \sin^2\alpha\, I_Y.$$

It may easily be proven that if two distances, a and b, be set off from O on X and Y respectively, made inversely proportional to $\sqrt{I_X}$ and $\sqrt{I_Y}$, and an ellipse be described on a and b as semi-axes; then the moments of inertia of the figure about any axes through O are inversely proportional to the squares of the corresponding semi-diameters of the ellipse, called therefore the Ellipse of Inertia. It follows that the moments of inertia about all gravity axes of a circle, or of any regular polygon, are equal, since their ellipses of inertia must be circles. Even if the plane figure is not symmetrical, an ellipse of inertia can be located at any point, and has the properties already mentioned. Its axes are called the principal axes for that point.

The Rectangle.—First, about its base. Fig. 20. Since all points of a strip parallel to the base have the same ordinate, we may take the area of such a strip as our element of area.

$$\text{Then } dA = bdz \text{ and } I_B = \int z^2 dA = b\int_0^h z^2 dz = \frac{1}{3} bh^3.$$

Second, about a gravity axis parallel to the base. Fig. 21.

$$dA = bdz, \text{ therefore } I_g = \int z^2 dA = b\int_{-\frac{h}{2}}^{+\frac{h}{2}} z^2 dz = \frac{1}{12} bh^3.$$

Third, about any other axis in its plane. Use the reduction formulæ.

The Triangle.—First, about an axis through the vertex parallel to the base. Fig. 22. Assume as the element of area a strip parallel to the base. Let its length be y, and its breadth dz. Then,

$$dA = ydz = \frac{b}{h} zdz.$$

$$\text{Therefore, } I_V = \int z^2 dA = \frac{b}{h}\int_0^h z^3 dz = \frac{1}{4} bh^3.$$

Second, about a gravity axis parallel to the base. Fig. 23. We

may obtain the desired result by applying the reduction formula for parallel axes to the value just obtained. Since $A=\frac{1}{2}bh$ and $d=\frac{2}{3}h$,

$$I_g=I_v-Ad^2=\frac{1}{4}bh^3-\frac{1}{2}bh.\frac{4}{9}h^2=\frac{1}{36}bh^3.$$

Third, about the base. $I_B=I_g+Ad^2$, with $d=\frac{1}{3}h$.

Therefore $I_B=\frac{1}{36}bh^3+\frac{1}{2}bh.\frac{1}{9}h^2=\frac{1}{12}bh^3.$

The results in the last two cases could have been obtained directly by integration without the use of the reduction formula.

The Circle.—About any diameter as an axis. Fig. 24. If we use polar co-ordinates, the element of area $dA=\rho d\phi d\rho$, and $z=\rho \sin\phi$.

$$I_g=\int z^2dA=\int\int(\rho\sin\phi)^2\rho d\phi d\rho=\int_0^{2\pi}\sin^2\phi d\phi\int_0^r\rho^3d\rho$$

$$=\frac{r^4}{4}\int_0^{2\pi}\sin^2\phi d\phi=\frac{r^4}{4}\int_0^{2\pi}\frac{1}{2}(1-\cos 2\phi)d\phi=\frac{\pi r^4}{4}.$$

Compound Plane Figures.—Since the moment of inertia of any plane figure about any axis is equal to the sum of the moments of inertia of its elements, it follows that the moment of inertia of any compound plane figure may be found by separating it into several simple plane figures whose moments of inertia about the given axis may readily be found and then taking the algebraic sum of the moments of inertia of these component parts.

The moment of inertia of the **T**, shown in Fig. 25, about an axis through the base is evidently equal to the moment of inertia of the circumscribed rectangle minus the sum of the moments of inertia of the two rectangles on either side, i. e., is equal to

$$\frac{1}{3}(bh^3-b_1{}^3h_1{}^3)$$

To find the moment of inertia of the same figure about the gravity axis parallel to the base, we would first find the distance from the base to the center of gravity and then apply the reduction formula for parallel axes to the result obtained above.

With this indication of the methods to be followed, the student is expected to be able to derive the moments of inertia of other compound figures, such as the I, hollow square, annulus, etc., about any axis.

In all of the preceding discussion of moments of inertia of plane figures, the axis in every case has been in the plane of the figure. Such moments of inertia are called *rectangular moments of inertia.*

Polar Moment of Inertia.—If the moment of inertia of a plane figure be obtained with reference to an axis perpendicular to the plane of the figure, the result is called a polar moment of inertia.

In Fig. 26, let Z be any axis perpendicular to the plane of the figure and let X and Y be two rectangular axes in its plane and having their origin in Z. Let ρ be the distance from Z to any element of area, dA. Then $I_Z = \int \rho^2 dA = \int (x^2+y^2)dA = I_X + I_Y.$

That is, the polar moment of inertia of any plane figure about any point in its plane is equal to the sum of the rectangular moments of inertia about any two rectangular axes in the plane which intersect at that point.

For the circle about its center, $I_Z = \frac{\pi r^4}{2}$.

For a rectangle about its center, $I_Z = \frac{1}{12} bh(b^2+h^2)$.

Thin Plates. Axis in Plate.—Let the plates be homogeneous and of constant, small thickness, t. Let the weight of the plate per unit volume be w, and its area, A; then its mass is $\frac{wAt}{g}$.

Now for a plate whose thickness is very small compared with its other dimensions, we may write,

$$I = \int \rho^2 dM = \frac{w}{g}\int \rho^2 dV = \frac{w}{g}t\int \rho^2 dA;$$

i. e., $= (w \div g) \times$ thickness $\times$ moment of inertia of the plane figure.

For a rectangular plate about a gravity axis parallel to the base, this reduces to $I_g = \frac{w}{g}t.\frac{1}{12}bh^3 = \frac{1}{12}Mh^2$.

The student will be expected to apply this principle to thin plates of other shapes.

Right Prisms of any Altitude about an Axis Perpendicular to the Base.—As before, the solid is considered homogeneous, its heaviness $= w$, and its altitude $= h$. Fig. 27. Consider an elementary prism whose length is parallel to the axis of reference, Z. The altitude of the element will be h, that of the whole solid, and its base will be dA, an element of the base, A, of the solid. Its mass $= \frac{whdA}{g}$.

The moment of inertia of the solid,

$I_z = \int \rho^2 dM = \frac{wh}{g}\int \rho^2 dA = \frac{wh}{g} \times$ the polar moment of inertia of the base.

Applying this to a circular right cylinder, radius of base $= r$, and altitude $= h$; we have for the moment of inertia about the axis of the cylinder, $I_g = \frac{wh}{g}.\frac{1}{2}\pi r^4 = \frac{1}{2}Mr^2$.

The student is expected to make other applications of this principle.

Homogeneous Solid Cylinder about a Diameter of its Base.—Consider the cylinder to be divided into an infinite

number of laminæ parallel to the base and of infinitesimal thickness. Let the distance of any lamina above the base be z, and its thickness be dz. Fig. 28. Then the moment of inertia of the lamina about its gravity axis parallel to the base of the cylinder is $\frac{w}{g}dz.\frac{\pi r^4}{4}$ and the moment of inertia of the lamina about the axis in the base of the cylinder is this quantity plus the mass of the lamina multiplied by the square of its distance above the base.

The moment of inertia of the cylinder is the sum of the moments of inertia of all the laminæ, i. e.,

$$
\begin{aligned}
I_X &= \int_0^h \left\{ \frac{w}{g} dz. \frac{\pi r^4}{4} + \frac{w}{g} dz.\, \pi r^2.z^2 \right\} \\
&= \frac{w\pi r^2}{g} \left\{ \frac{1}{4} r^2 \int_0^h dz + \int_0^b z^2 dz \right\} \\
&= \frac{w\pi r^2 h}{g} \left\{ \frac{r^2}{4} + \frac{h^2}{3} \right\} = M \left\{ \frac{r^2}{4} + \frac{h^2}{3} \right\}.
\end{aligned}
$$

Homogeneous Right Cone.—*First,* about an axis through the apex and parallel to the base. Fig. 29. Consider the cone divided into laminæ parallel to the base. Then, if x represents the radius of any lamina at a distance, z, from the apex, $x = \frac{r}{h} z$. The moment of inertia of any lamina about the assumed axis will be $\frac{\pi x^2 w dz}{g} \left\{ \frac{1}{4} x^2 + z^2 \right\}$ and the moment of inertia of the whole cone will be $I_A = \int_0^h \frac{\pi x^2 w dz}{g} \left\{ \frac{1}{4} x^2 + z^2 \right\}$

$$
= \frac{w\pi r^2}{gh^2} \left\{ \frac{1}{4} . \frac{r^2}{h^2} + 1 \right\} \int_0^h z^4 dz = \frac{3}{20} M (r^2 + 4h^2).
$$

Second, about a gravity axis parallel to the base. By use of the reduction formula for parallel axes and the result just obtained, we have

$$
I_g = \frac{3}{20} M (r^2 + \frac{h^2}{4}).
$$

Third, about its geometric axis. Since the axis is perpendicular to each circular lamina, the moment of inertia of any lamina is equal to its mass $\times \frac{1}{2}(\text{radius})^2 = \frac{w\pi x^2 dz}{g} \cdot \frac{x^2}{2}$.

Now $x = \frac{r}{h} z$, hence for the whole cone

$$I_z = \int_0^h \frac{w\pi r^4}{2gh^4} z^4 dz = \frac{w\pi r^2 h}{g} \cdot \frac{1}{10} r^2 = \frac{3}{10} Mr^2.$$

Homogeneous Right Pyramid with Rectangular Base.—About its geometric axis. Proceeding as in the last case, we obtain $I_g = \frac{Md^2}{20}$, d being the diagonal of the base.

Homogeneous Sphere.—About any diameter. Fig. 30. Divide into laminæ perpendicular to the axis. Noting that $x^2 = r^2 - z^2$, we have for the whole sphere

$$I_z = \frac{w\pi}{2g} \int x^4 dz = \frac{w\pi}{2g} \int_{-r}^{+r} (r^4 - 2r^2z^2 + z^4) dz$$

$$= \frac{w\pi}{2g} \left[(r^4 z - \frac{2}{3} r^2 z^3 + \frac{z^5}{5}) \right]_{-r}^{+r} = \frac{8}{15} \cdot \frac{w\pi r^5}{g} = \frac{2}{5} Mr^2.$$

For a segment of one or two bases put proper limits for z in above instead of $+r$ and $-r$.

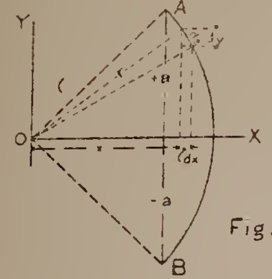


Fig. 1.

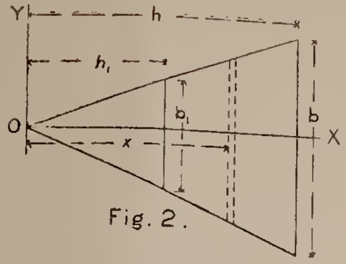


Fig. 2.

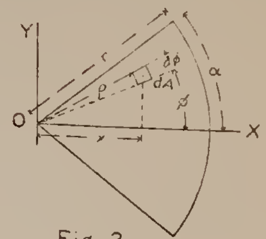


Fig 3.

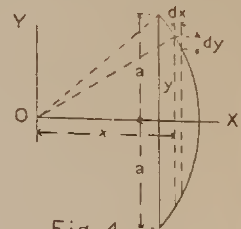
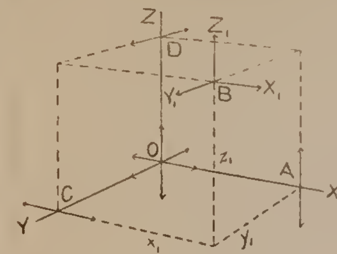


Fig. 4.



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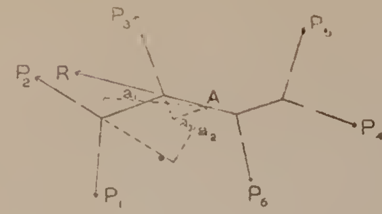

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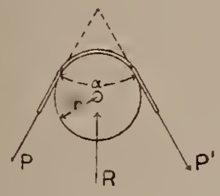

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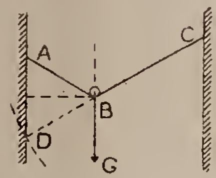

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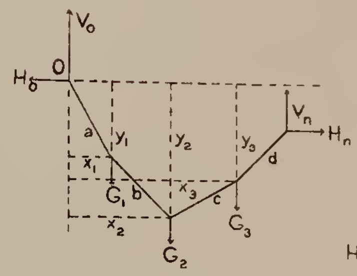


Fig. 9.

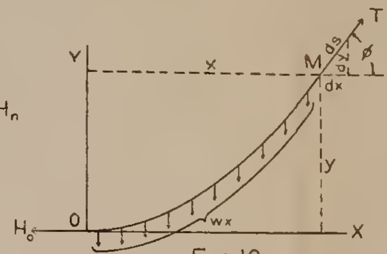
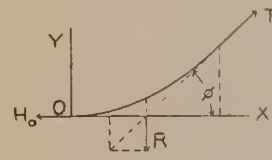


Fig. 10.



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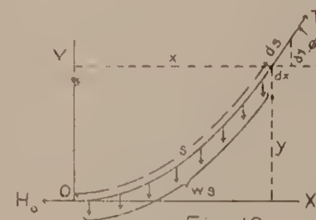

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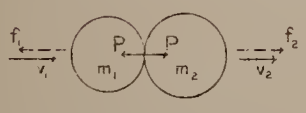


Fig. 13.

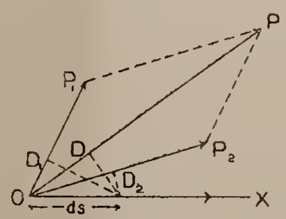


Fig. 14.

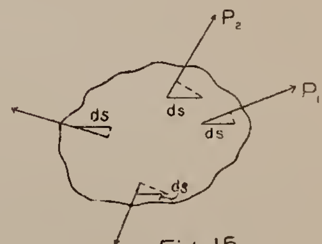
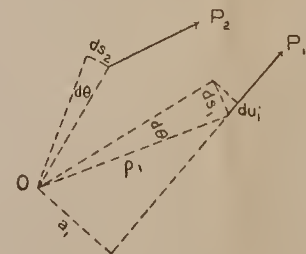


Fig 15.



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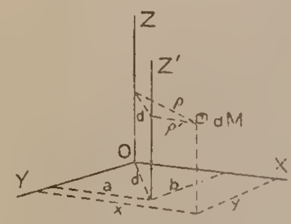
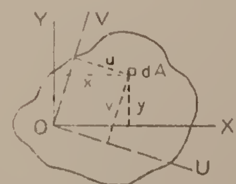


Fig. 17.



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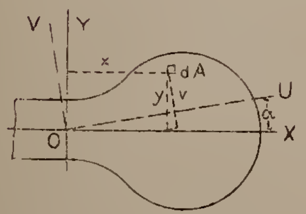
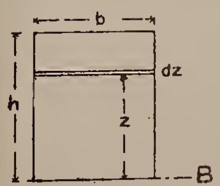


Fig. 19.



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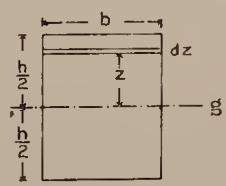
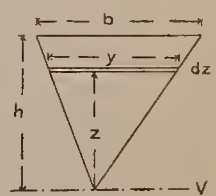


Fig. 21.



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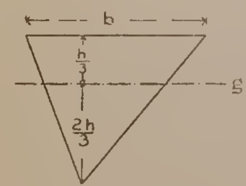

Fig. 23.

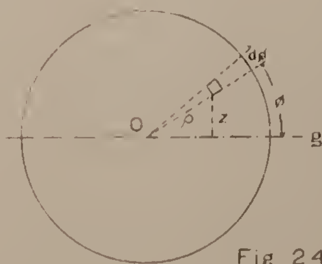

Fig 24.

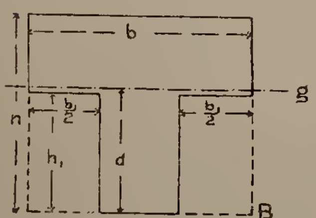

Fig. 25.

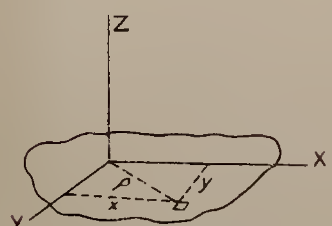

Fig. 26.

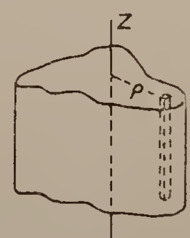

Fig. 27.

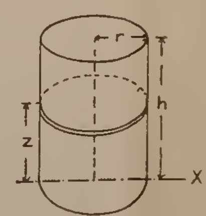

Fig. 28.

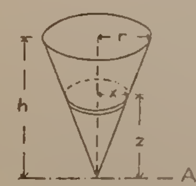

Fig. 29.

Fig. 30.